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# On the Stochastic Approximation Method and Optimal Filtering Theory

YU CHI HO*

*Harvard University, Cambridge, Massachusetts and The RAND Corporation, Santa Monica, California*

*Submitted by Richard Bellman*

In this short paper we wish to establish some connections among the maximum likelihood estimate, the optimal filtering, and the stochastic approximation solutions to the following well-known problem: Consider the vector-matrix equations

$$Ax + v_k = b_k \qquad k = 1, 2, \cdots \tag{1}$$

where

$$A \text{ is a given } r \times n \text{ matrix}$$
$$x \text{ is an unknown } n\text{-vector}$$
$$v_k \text{ is a random } r\text{-vector with } E(v_k) = 0$$

and

$$E(v_k v_j') = I\delta(k - j)$$
$$b_k \text{ is a } r\text{-vector of observation}$$

One wishes to determine an estimate $x$ for the unknown parameters $x$ which is optimal in some sense.

First, we shall make a maximum likelihood estimate for $x$. It is well-known [1, 2] that a recursive method for calculating the estimate in the case of gaussian noise is

$$\hat{x}_{k+1} = \hat{x}_k + P_{k+1}A'(b_{k+1} - A\hat{x}_k) \tag{2}$$
$$\hat{x}_0 = 0$$

where

$$P_{k+1}^{-1} = P_k^{-1} + A'A \tag{3}$$
$$P_0 = \text{given } n \times n \text{ positive definite matrix}$$

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Note that (3) only yields the inverse of $P_{k+1}$ which is needed in (2). However, it is well known in numerical analysis, though not in control theory, that the following lemma is true [6]:

LEMMA. *If* $P_{k+1}^{-1} = P_k^{-1} + A'A$ *where* $P_k > 0$ *then*

$$P_{k+1} = P_k - P_kA'(AP_kA' + I)^{-1} AP_k \tag{4}$$

PROOF: by direct verification. Using (4) one can easily establish again by straightforward matrix manipulation that

$$P_{k+1}A' = P_kA'(AP_kA' + I)^{-1} \tag{5}$$

which leads to

$$\hat{x}_{k+1} = \hat{x}_k + P_kA'(AP_kA' + I)^{-1} (b_{k+1} - A\hat{x}_k) \tag{6}$$

Equations (4) and (6) are essentially the estimation scheme of Kalman based on the original Wiener's theory [3]. Bellman [4] has also derived (4) using dynamic programming.

Now repeated applications of (5) to itself lead to

$$\begin{aligned} P_kA' &= P_{k-1}A'(AP_{k-1}A' + I)^{-1} = P_{k-2}A'(2AP_{k-2}A' + I)^{-1} \\ &= P_0A'(kAP_0A' + I)^{-1} \end{aligned} \tag{7}$$

such that we have in the limit

$$\lim_{k\to\infty} P_kA' = \frac{1}{k} P_0A'(AP_0A')^{-1} \tag{8}$$

Thus, (6) becomes asymptotically

$$\hat{x}_{k+1} = \hat{x}_k + \frac{1}{k+1} P_0A'(AP_0A')^{-1} (b_{k+1} - A\hat{x}_k) \tag{9}$$

We recognize that (9) represents a form of multidimensional stochastic approximation [5] with the weighting function going down as $1/k$ as it is required. Furthermore, if we take $r = n$ (which represents no loss of generality inasmuch as we are investigating the case where $k \to \infty$) and assume that $A^{-1}$ exist (which is the same as the usual observability condition [3]), then (9) becomes simply

$$\begin{aligned} \hat{x}_{k+1} &= \hat{x}_k + \frac{1}{k+1} [x - (\hat{x}_k + A^{-1}v_k)] \\ &= \frac{1}{k+1} \sum_{i=0}^{k+1} (x - A^{-1}v_i) = \frac{1}{k+1} \sum_{i=0}^{k+1} A^{-1}b_i \end{aligned} \tag{10}$$

which is just the weak law of large numbers.

Equations (4), (5), and (7) can be extended in many ways to include effect of time dependence, etc. The expressions become, however, more involved. For example, for $r = n$, (7) becomes

$$P_k A_k' = P_0 A_1' \left[ \left( \sum_{i=1}^{k} A_1^{-1} (A_i' A_i) \right) P_0 A_1' + I \right]^{-1} A_1^{-1\prime} A_k' \tag{11}$$

for the case where $A$ is time varying. Note that the $1/k$ effect is nevertheless still present. The presence of $P_0$ in the formula allows us to take advantage of the confidence one has in the initial estimate $\hat{x}_0$. While in the regular stochastic approximation scheme, this fact is missing and the weighting factor goes down as $1/k$ instead of $1/k + k_0$ where $k_0$ is a constant.

## References


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4. Bellman, R. "Introduction to Matrix Analysis," pp. 154-155. McGraw Hill, New York, 1960.
5. Blum, J. Multidimensional stochastic approximation method. *Ann. Math. Statist.* **25**, 737-743 (1954).
6. Bodewig, E. "Matrix Calculus," p. 30. Interscience, New York, 1956.


[1] The author wishes to acknowledge the earlier discovery of the connection between the maximum likelihood method and the Wiener-Kalman method by Bryson and Frazier. They have derived (3) from (4) without using the lemma in an earlier unpublished note which was brought to the author's attention after the writing of the paper. Further extension and generalization may be found in their recent paper "Smoothing for linear and nonlinear dynamic systems" to be published in the Proceedings of the Systems Optimization Conference, Wright-Patterson Field, Ohio, September 1962.